

Collective Current Rectification

S. Denisov

*Institute of Physics, University of Augsburg, Universitätsstrasse 1, D-86159
Augsburg, Germany*

Abstract

We consider a set of coupled underdamped ac-driven dynamical units exposed to a heat bath. The coupling scheme defines the absence/presence of certain symmetries, which in turn cause a nonzero/zero value of a mean dc-output. We discuss dynamical mechanisms of a dc-current appearance and identify current reversals with synchronization/desynchronization transitions in the collective ratchet's dynamics.

Key words:

driven underdamped systems; synchronization; ratchet effect

PACS: 05.45.ac, 05.60.cd

1 Introduction

The *ratchet* effect, i. e. the possibility to obtain the directed transport by using zero-mean perturbations only, has induced notable scrutiny over the last decade [1]. Initially, most studies have been focused on noisy overdamped models have been inspired by a molecular motors realm [1]. Then the ratchet's approach has been applied to a broad class of physical systems in which inertia effects are essential [2]. The examples are Josepshon junctions [3], cold atoms systems [4], and mechanical engines [5].

Recently it has been shown that the ratchet idea can be viewed as a part of a general symmetry-breaking approach [6]. This approach is based on the analysis of all relevant symmetries which have to be broken in order to fulfill necessary conditions for a dc-output appearance. The formalization of symmetry analysis for one particle's dynamics has been addressed in Ref.[7].

Email address: sergey.denisov@physik.uni-augsburg.de (S. Denisov).

In the present paper we aim at collective rectification effects which arise in a set of coupled single ratchet units. Although various examples of interacting ratchet systems have been proposed already in the context of molecular motors [8], here the emphasis is put on the weak-noise underdamped limit. We show that a coupling scheme determines a set of certain symmetries for the ratchet's collective and hence defines necessary conditions for a dc-current generation. Dynamical mechanisms of a current rectification are connected with a coherence between units, which depends not only on a coupling scheme, but also on a strength of interactions.

2 Coupling schemes and symmetries

Let us consider a set consisting of N identical dynamical units, $\mathbf{x} = \{x_i, i = 1, \dots, N\}$, that are linearly and symmetrically coupled. The coupling scheme is described by some graph which can be encoded in the symmetrical $N \times N$ binary matrix G_{ij} , $G_{ij} = G_{ji}$. The equations of motion are the following:

$$\ddot{x}_i = -\alpha \dot{x}_i + F(x_i - x_i^0, t - t_i^0) + c \sum_{j=1}^N G_{ij} H(x_i, x_j, \dot{x}_i, \dot{x}_j) + \xi_i(t), \quad (1)$$

where $F(x+L, t) = F(x, t+T) = F(x, t)$ is the double periodic force function, H is linear over all arguments coupling function, and c is the strength of interactions. The stochastic terms $\xi_i(t)$ are mutually independent delta-correlated Gaussian white noises, $\langle \xi_i(t) \xi_j(s) \rangle = \sigma \delta_{ij} \delta(t-s)$, where σ is the noise intensity. We assume also that the force function, F , is the same for all the units, but the phases, x_i^0 and t_i^0 , can be different for different units. Finally, we are interested in the mean dc-current,

$$J = \lim_{t \rightarrow \infty} \frac{1}{Nt} \sum_{i=1}^N x_i(t). \quad (2)$$

Following the symmetry analysis ideology [6-7], in order to determine necessary conditions for a dc-output appearance, we have to check whether there exist symmetry transformations which allow to generate out for each trajectory of the system (1) another one with a reversal velocity. The presence of the white noise, $\xi_i(t)$, does not change the symmetry properties of the system. Moreover, the coupling to a heat bath leads to an effective exploration of the whole phase space and produce an averaging in a case of several coexisting attractors [6,9].

For the one-particle case, $N = 1$, a transformation of interest has to involve a change of the sign of x (and thus change of the current direction). It allows

also some shifts in time and space domains [6-7]:

$$\hat{S}_{single} : x \rightarrow -x + \lambda, \quad t \rightarrow t + \tau, \quad x \in \mathbf{R}. \quad (3)$$

In the case of several coupled ratchets a symmetry operation should be performed in the global coordinate space \mathbf{R}^N and can also involve a permutation between different units. The corresponding symmetry operation can be described as the linear transformation,

$$\hat{S}_{network} : \mathbf{x} \rightarrow -\mathbf{S}\mathbf{x} + \lambda, \quad t \rightarrow t + \tau, \quad \mathbf{x} \in \mathbf{R}^N, \quad (4)$$

where S_{ij} is a $N \times N$ binary matrix with only one non-zero element at each row and line, and $\lambda = \{\lambda_i\}, i = 1, \dots, N$ is the vector of shifts. The matrix \mathbf{S} encodes permutation between units.

The properties of the permutation matrix S_{ij} (and the existence of such a matrix at all) are strongly depend on the structure of connections among units. Below, using simple examples, we will illustrate this statement.

Breaking symmetries by connections. Let us first consider the case of two particles (rotators) in the standing-wave potential with the modulated amplitude [10], coupled by the linear spring:

$$\ddot{x}_1 = -\alpha\dot{x}_1 + F(x_1, t) + c(x_2 - x_1) + \xi_1(t) \quad (5)$$

$$\ddot{x}_2 = -\alpha\dot{x}_2 + F(x_2 - x_0, t - t_0) + c(x_1 - x_2) + \xi_2(t), \quad (6)$$

where $F(x, t) = \sin(x) \sin(\omega t)$.

In the uncoupled case, $c = 0$, both the systems posses symmetries of the type (3):

$$\hat{S}_1 : x \rightarrow -x + \pi, t \rightarrow t + T/2 \quad (7)$$

$$\hat{S}_2 : x \rightarrow -x + \pi + x_0, t \rightarrow t + T/2 + t_0, \quad (8)$$

with $x_0 = t_0 = 0$ for the first rotator, Eq.(5). The symmetry transformations, Eqs.(7-8), are independent for each rotators. The whole system, Eqs.(5-6), is symmetric with respect to the transformation $\hat{S}_1 \times \hat{S}_2$ for any choice of t_0 and x_0 . So, the mean dc-output for the uncoupled case is zero (line(1) in Fig.1a).

In the case of coupled particles, $c > 0$, abovementioned transformations, , Eqs.(7-8), should been conjugated and independent spatial and temporal shifts are forbidden now. For $x_0 \neq k\pi$ and $t_0 \neq T/2$, the connection breaks both the symmetries and we can expect on nonzero current appearance (see also Ref.

[11] for another example of an overdamped dimer with an additive driving force). For the set of parameters $\alpha = 0.1$, $w = 0.3$, $x_0 = \pi/2$, $\sigma = 0.01$ and $t_0 = T/4$, the symmetry violation is realized by the asymmetrical limit cycle with the negative winding number (line(2) in Fig.1a). So, in this case the connection between the units destroy all the symmetries and leads to the dc-current generation.

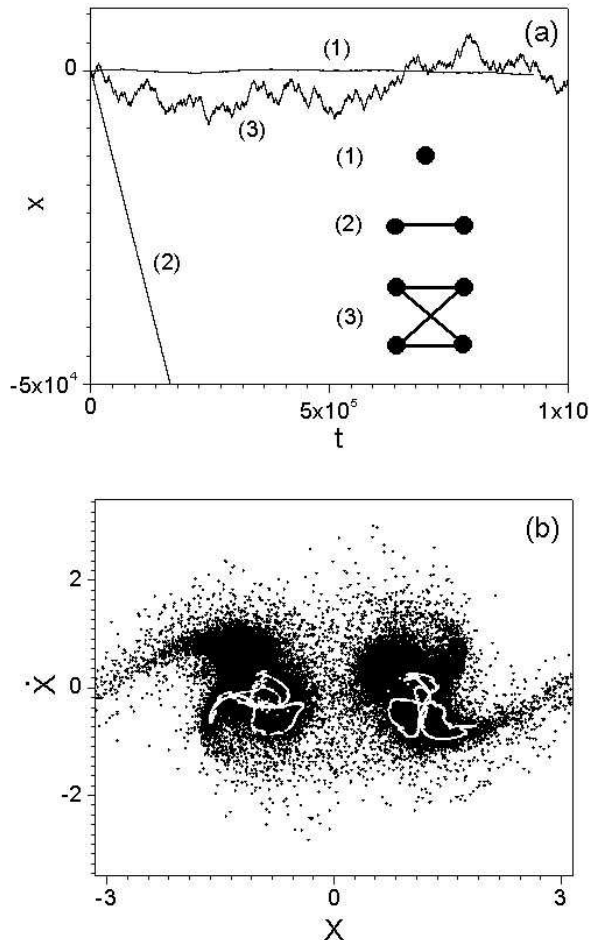


Fig. 1. (a) The dependence of $x_1(t)$ versus t for different coupling schemes (see text for details) for the parameter values $\alpha = 0.1$, $w = 0.3$, $x_0 = \pi/2$ and $t_0 = T/4$ and noise intensity $\sigma = 0.01$. The dependencies for another rotators from the set are the same due to the homogeneity of the system, Eqs.(5-6) and Eqs.(9-10); (b) Poincaré sections for the first rotator for the third variant of the coupling scheme. The coupling to the heat bath leads to the averaging over two symmetry-related chaotic attractors (white dots) with opposite mean velocities.

Restoring symmetries by connections. Let us now consider the dimer identical to the previous one, Eq.(5-6), but spatially shifted by π (half of the period):

$$\ddot{x}_3 = -\alpha \dot{x}_3 + F(x_3 + \pi, t) + c(x_4 - x_3) + \xi_3(t) \quad (9)$$

$$\ddot{x}_4 = -\alpha\dot{x}_4 + F(x_2 - x_0 + \pi, t - t_0) + c(x_3 - x_4) + \xi_4(t). \quad (10)$$

Both the systems, Eqs.(5-6) and Eqs.(9-10), can be transformed from one to another by the simple coordinate shift. Thus, the system in Eqs.(9-10) produces the same mean current, as the system in Eqs.(6-7).

As the next step we couple both the systems by additional connections ((1) \leftrightarrow (4), (2) \leftrightarrow (3)) (see the inset in Fig.1a). The generalized symmetry transformation, Eq.(4), can be identified with the following permutation matrix:

$$S_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (11)$$

and the shift vector $\lambda = \{\pi\}$.

Here, despite to the previous case, the introduction of additional connections leads to the appearance of the new symmetry and we may expect on the nullification of the dc-output (line(3) in Fig.1a). For the above set of parameters we found that the symmetry is realized in the phase space by two symmetry-coupled chaotic attractors with opposite mean velocities (see Fig.1b). It is easy to check, that any other type of connection with two additional links does not restore the system's symmetry.

3 Synchronization and current reversals

While the presence or the absence of the dc-output is clearly connected to the absence/presence of the symmetry transformation, the dc-current value is determined by dynamical mechanisms. From the previous studies [12] it is known that global properties of a collective dynamics are defined by a coherence between units. Thus, we can expect that an efficiency of the current rectification is closely related to a degree of synchronization within a ratchet's collective.

As a natural example we consider the model of N globally coupled array of underdamped Josephson junctions (JJ), subjected to an ac-current $E(t)$ [13]. The equation for the superconducting phase difference x_i across the single junction is

$$\ddot{x}_i = -\alpha\dot{x}_i + \sin(x_i) + E(t) + c(\dot{x}_i - \langle \dot{x} \rangle_A) + \xi_i(t), \quad (12)$$

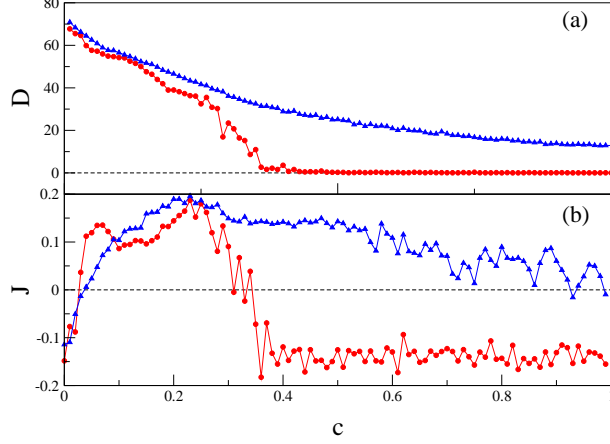


Fig. 2. The dependence of the mean dc-output, J , and the degree of decoherence, D , versus the strength of interaction c for the system of $N = 50$ coupled units from Eq.(12). The values of parameters are $\alpha = 0.1$, $w = 1$, $\sigma = 0$ (circles) and $\sigma = 0.03$ (triangles).

where $\langle \dot{x} \rangle_A = \frac{1}{N} \sum_{i=1}^N \dot{x}_i(t)$ is the instantaneous mean array current. As the driving force we used the two-harmonics combination, $E(t) = \cos(\omega t) + \cos(\omega t + \pi/2)$, which ensures that all the relevant symmetries are broken [7]. The numerically obtained dependence of the array dc-output on the strength of interaction c is shown in Fig.2(a). For the set of parameters $\alpha = 0.1, \omega = 1, \sigma = 0$, the dependence demonstrates the presence of two successive current reversals, at $c \approx 0.03$ and $c \approx 0.33$. In order to understand dynamical mechanisms of these events, we introduced the mean decoherence, which we define as

$$D = \langle |x(t) - \langle \dot{x} \rangle_A| \rangle_{A,T}, \quad (13)$$

where $\langle \dots \rangle_{A,T}$ means the averaging over the array and over the one period of ac-driving. The first result from the comparison of both the dependencies (Fig.2(b)) is that the current reversal at $c \approx 0.33$ is connected with the transition to the regime of the complete synchronization, $D = 0$.

Lets now track briefly a relation between the current reversal and the synchronization. Initially, in the uncoupled limit, $c = 0$, all JJ's work independently, each one as the single one-dimensional rectifier [1,6]. The asymmetry of the ac-force $E(t)$ is realized through the chaotic attractor with the positive mean current (see Fig.3a) [14]. The local phase space of each units, (x_i, \dot{x}_i, t) , has identical structure, but the relative phases of different units are randomly distributed (depending on the initial conditions).

After the introduction of a nonzero coupling, $c > 0$, some coherence between the units occurs. From a point of view of the dynamics of the single junction, this leads to changes of an attractor structure in the phase space \mathbf{R}^3 (see

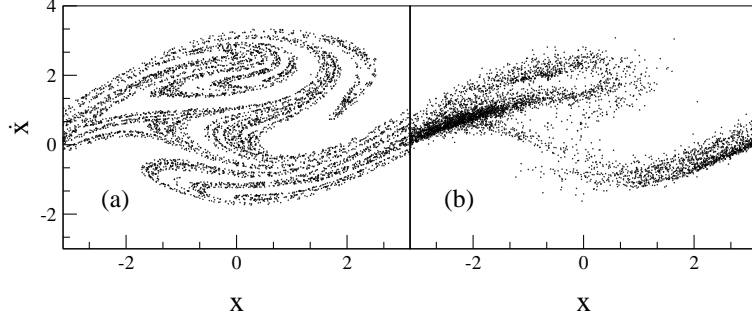


Fig. 3. Poincaré sections for a single junction from the array of $N = 50$ coupled JJ, Eq.(12), for (a) $c = 0$ and (b) $c = 0.27$ (zero-noise case). Parameter values are the same as in Fig.2.

Fig.3b). This causes changes in a projection of the attractor's invariant density on the velocity subspace \dot{x}_i and, as a result, leads to the current reversal at $c \approx 0.03$. The further increase of the interaction strength up to $c \approx 0.33$ results in the complete synchronization and to a shrinking of the global attractor in \mathbf{R}^{2N+1} to the hyperplane $\mathbf{R}^3, (x_i = x, \dot{x}_i = \dot{x}, t)$. The global attractor has now dimension equal to 3. The attractor of the single unit now is the same as in the uncoupled limit but all the units have the same phase. The mean dc-output returns to its value at the limit $c = 0$.

Here some analogy with a current reversal in an one-dimensional deterministic ratchet [14] can be drawn. In the one-dimensional case current reversals have been identified with tangent bifurcations from chaotic to regular (limit cycles) attractors [14]. In our case the current reversal at $c \approx 0.33$ corresponds to the transition "hyperchaos - chaos" [15], connected with the shrinking of a the system attractor in a global phase space \mathbf{R}^{2N+1} . It is interesting to note, that from a point of view of the local attractor of the single junction, this transition corresponds to a crisis, connected with *expanding* of the attractor in three-dimensional subspace (x_i, \dot{x}_i, t) (compare Fig.3a and Fig.3b).

The presence of a weak noise (which is equivalent to a weak coupling with a heat bath) can strongly suppress correlations between units and may lead to a delayed synchronization transition [16]. Due to a strong conjugation between the synchronization and the current rectification process, we can manage a noise-induced current reversal for a fixed strength of interaction (Fig.2(b)). Thus, the presence of a thermostat allows to control the dc-output by changing a temperature of the system.

4 Concluding remarks

Finally, we have presented the symmetry approach to the problem of the collective current rectification by a set of coupled dynamical units. This idea can be used for a more general problem such as an obtaining of a non-zero value of some relevant mean ensemble characteristic, $\langle A(\mathbf{x}) \rangle_t$. This characteristic can correspond, for example, to a mean magnetization of a spin lattice with a complex geometry [17]. The proposed collective ratchet's ideology may also be relevant in a context of a cooperative dynamics of neural networks, as an approach to a visual processing with directional selectivity [18].

Relations between symmetries of a collective ratchet and its coupling scheme on the one side, and relations between a topology of interactions and synchronization properties [19], on the other one, may open an interesting perspective. It has been found that in the thermodynamic limit, $N \rightarrow \infty$, the synchronization is impossible for nearest-neighbor coupled dynamical networks if the number of sites connected to a given site, N_c , is a finite fraction of the total number of sites N , $\frac{N_c}{N} = \text{const}$ [20]. On the other hand, in a case of small-world networks, i.e. sets with long (in a sense of a topological distance) connections, the synchronization can be achieved in the thermodynamic limit through a small fraction of distant connections [20]. This nontrivial effect provides a tool for a control of the dc-output in a massive collective ratchet by changing a small number of relevant connections only.

This work has been supported by the Emmy Noether-Programm of the DFG under contract LU1382/1-1

References

- [1] P. Hänggi and R. Bartusek, Lect. Notes. Phys. 476, (1996) 294; F. Julicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. 69 (1997) 1269; R. D. Astumian and P. Hänggi, Physics Today 55 (2002) 33; P. Reimann, Phys. Rep. 361 (2002) 57.
- [2] P. Jung, J. G. Kissner, and P. Hänggi, Phys. Rev. Lett. 76 (1996) 3436; M. Barbi and M. Salerno, Phys. Rev. E 62 (2000) 1988; L. Machura et al, Phys. Rev. E 70 (2004) 061105; L. Machura et al, J. Phys. C: Condens. Matter 17 (2005) S3741.
- [3] G. Carapella, Phys. Rev. B 63 (2001) 054515; G. Carapella and G. Costabile, Phys. Rev. Lett. 87 (2001) 077002.
- [4] M. Schiavoni, L. Sanchez-Palencia, F. Renzoni, and G. Grynberg, Phys. Rev. Lett. 90 (2003) 094101; P. H. Jones, M. Goonasekera, and F. Renzoni, Phys. Rev. Lett. 93 (2004) 073904; R. Gommers, S. Denisov, and F. Renzoni, Phys. Rev. Lett. 96 (2006) 240604.

- [5] B. Norden, Y. Zolotaryuk, P. L. Christiansen, and A. V. Zolotaryuk, Phys. Rev. E 65 (2002) 011110.
- [6] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. Lett. 84 (2000) 2358.
- [7] S. Denisov *et al*, Phys. Rev. E. 66 (2002) 041104.
- [8] F. Jülicher and J. Prost, Phys. Rev. Lett. 75 (1995) 2618; I. Derényi and T. Vicsek, Phys. Rev. Lett. 75 (1995) 374; P. Reimann, R. Kawai, C. Van den Broeck, and P. Hänggi, Europhys. Lett. 45 (1999) 545.
- [9] J. L. Mateos, Physica A 325 (2003) 92.
- [10] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, Nonlinear physics : from the pendulum to turbulence and chaos, Harwood Academic Publ., 1992.
- [11] S. Cilla, F. Falo, and L. M. Floria, Phys. Rev. E 63 (2001) 031110.
- [12] V. S. Afraimovich, V. I. Nekorkin, G. V. Osipov, and V. D. Shalfeev, Stability, Structures and Chaos in Nonlinear Synchronization Network, World Scientific, Singapore, 1994.
- [13] K. Wiesenfeld and P. Hadley, Phys. Rev. Lett. 62 (1989) 1335; K. Y. Tsang et al., *ibid.* 66 (1991) 1094; K. Y. Tsang and I. B. Schwartz, *ibid.* 68 (1992) 2265.
- [14] J. L. Mateos, Phys. Rev. Lett. 84 (2000) 258; J. L. Mateos, Physica D, 168-169 (2002) 205.
- [15] E. Ott, Chaos in Dynamical Systems, Cambridge University Press, Cambridge, 1992.
- [16] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: Universal Concept in Nonlinear Science, Cambridge University Press, Cambridge, 2001.
- [17] D. C. Cabra and M. D. Grynberg, Phys. Rev. Lett. 82 (1999) 1768, Ph. Kurz, G. Bihlmayer, K. Hirai, and S. Blugel, *ibid.* 86 (2001) 1106;
- [18] H. Sompolsky and D. Golomb, Phys. Rev. A 43 (1991) 6990.
- [19] U. E. Vincent, A. N. Njah, O. Akinlade, and A. R. T. Solarin, Chaos 14 (2004) 1018; M. Kostur, P. Hänggi, P. Talkner, and J. L. Mateos, Phys. Rev. E 72 (2005) 036210.
- [20] X. F. Wang, Int. J. Bif. Chaos 12 (2002) 885; B. Ao and Z. Zheng, Europhys. Lett. 74 (2006) 229.